NATIONAL MATH + SCIENCE INITIATIVE

BC students need to be familiar with the tests for convergence that are listed on the next page. They need to recognize various types of numerical series and efficiently apply the appropriate test. Most commonly used tests on the AP exam are the p-Series Test, the Geometric Series Test, the Alternating Series Test, and the nth-Term Test. Less common are the Integral Test and the Direct or Limit Comparison Tests. Students should also know how to find the sum of an infinite geometric series be able to use that sum to create a power series and determine its interval of convergence. Students should also be able to use substitution, differentiation and integration to create series related to given series. Finally, students should be able to use the Ratio Test to determine radius or open interval of convergence of power series. After using the Ratio Test to find the radius or open interval of convergence, they would then use the other tests (usually Alternating Series, p-Series, or nth Term) to check convergence at the endpoints.

While some students are intimidated by the series questions on the BC exam, these questions are somewhat predictable and provide an excellent opportunity to earn points that other students will not.

Test	Series	Converges	Diverges	Notes
n^{th} term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Geometric Series	$\sum_{n=1}^{\infty} a \cdot r^{n-1}$	$-1 < r < 1$	$r \le -1$ *or* $r \ge 1$	Sum= $\frac{a}{1-r}$
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \le 1$	
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$	*No*	*Yes*	The harmonic series is a p-series with $p = 1$.
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$0 < a_{n+1} < a_n$ *and* $\lim_{n\to\infty} a_n = 0$		$\lvert\text{Error}\rvert \le a_{n+1}$
Integral Test	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \ge 0$	*if* $\int_1^{\infty} f(x)dx$ converges	*if* $\int_1^{\infty} f(x)dx$ diverges	$f(x)$ is continuous, positive and decreasing.
Limit Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *and* $\sum_{n=1}^{\infty} b_n$ *converges*	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges*	Choose b_n that "behaves like" a_n.
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n < b_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges*	$0 < b_n < a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges*	
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive when $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$

Multiple Choice Questions: Unless otherwise indicated, all questions are no-calc.

1. (1973 BC19)
   Which of the following series converge?

   I. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   II. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

   III. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   (A) I only
   (B) III only
   (C) I and II only
   (D) I and III only
   (E) I, II, and III

2. (1985 BC14)
   Which of the following series are convergent?

   I. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$

   II. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$

   III. $1-\frac{1}{3}+\frac{1}{3^2}-\ldots+\frac{(-1)^{n+1}}{3^{n-1}}+\ldots$

   (A) I only
   (B) III only
   (C) I and III only
   (D) II and III only
   (E) I, II, and I

3. (1985 BC31)

   What are all values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

   (A) $-1 \le x < 1$
   (B) $-1 \le x \le 1$
   (C) $0 < x < 2$
   (D) $0 \le x < 2$
   (E) $0 \le x \le 2$

4. (1993 BC16)
Which of the following series diverge?

I. $\displaystyle\sum_{k=3}^{\infty} \frac{2}{k^2+1}$

II. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k$

III. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k}$

(A) None
(B) II only
(C) III only
(D) I and III only
(E) II and III only

5. (1993 BC27)
The interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ is

(A) $-3 < x \le 3$
(B) $-3 \le x \le 3$
(C) $-2 < x < 4$
(D) $-2 \le x < 4$
(E) $0 \le x \le 2$

6. (1997 BC14)
The sum of the infinite geometric series $\dfrac{3}{2} + \dfrac{9}{16} + \dfrac{27}{128} + \dfrac{81}{1{,}024} + \ldots$ is

(A) 1.60
(B) 2.35
(C) 2.40
(D) 2.45
(E) 2.50

7. (1997 BC20)

What are all values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$ converges?

(A) $-3 < x \le 3$
(B) $-3 < x < 3$
(C) $-1 < x \le 5$
(D) $-1 \le x \le 5$
(E) $-1 \le x < 5$

8. (1998 BC18)
Which of the following series converge?

I. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(A) None
(B) II only
(C) III only
(D) I and II only
(E) I and III only

9. (1998 BC22)

If $\displaystyle\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges

(E) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges

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10. (1998 BC76 - calc permitted)

For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

(A) 6
(B) 5
(C) 4
(D) 3
(E) 2

11. (1998 BC84 - calc permitted)

What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

(A) $-3 < x < -1$
(B) $-3 \le x < -1$
(C) $-3 \le x \le -1$
(D) $-1 \le x < 1$
(E) $-1 \le x \le 1$

12. (2003 BC10)

What is the value of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$?

(A) 1
(B) 2
(C) 4
(D) 6
(E) The series diverges

13. (2003 BC22)

What are all values of p for which the infinite series $\sum_{n=1}^{\infty} \frac{n}{n^p + 1}$ converges?

(A) $p > 0$
(B) $p \ge 1$
(C) $p > 1$
(D) $p \ge 2$
(E) $p > 2$

14. (2003 BC24)

Which of the following series diverge?

I. $$\sum_{n=0}^{\infty}\left(\frac{\sin 2}{\pi}\right)^{n}$$

II. $$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$$

III. $$\sum_{n=1}^{\infty}\left(\frac{e^{n}}{e^{n}+1}\right)$$

(A) III only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II, and III

Free Response Questions:

2001 BC6 - no calc – mean score 3.68/9

A function f is defined by

$$f(x)=\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\ldots+\frac{n+1}{3^{n+1}}x^n+\ldots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

2002 BC6 - no calc – mean score 4.00/9

The Maclaurin series for the function f is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\ldots+\frac{(2x)^{n+1}}{n+1}+\ldots$$

on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

2006 BC6 - no calc – mean score 3.31/9

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \ldots + \frac{(-1)^n nx^n}{n+1} + \ldots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^n x^n}{(2n)!} + \ldots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer

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2008B BC6 - no calc – mean score 3.84/9

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\dfrac{5}{4}\right)\right| < \dfrac{1}{100}$. Justify your answer.

2002B BC6 - no calc – mean score 4.22/9

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

2010B BC6 - no calc– mean score 2.00/9

The Maclaurin series for the function f is given by $f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Macluarin series of f. Justify your answer.

(b) Show that $y=f(x)$ is a solution to the differential equation $xy'-y=\frac{4x^2}{1+2x}$ for $|x|<R$, where R is the radius of convergence from part (a).